Highest weight modules and Verma modules

Binhe Huang

0. Notation.

- L fin dim semisimple over a algebraically closed field F of char 0.
- $L = H \oplus N \oplus N^-$, H Cartan subalg
- Φ root system, $\Delta = \{\alpha_1, \alpha_2, \cdots, \alpha_\ell\}$ the set of simple roots
- V a rep of L. $V_\lambda = \{ v \in L : h \cdot v = \lambda(h) v, \forall h \in H \}$, $\lambda \in H^*$ is called weight
  (V_λ: weight space)

I. Weight Spaces.

Thm. If V finite dim, $V = \bigoplus_{\lambda \in H^*} V_\lambda$. But it may not be true for inf. dim.

Ex. Consider $V = F[x]$ as a inf. dim sl_2-mod,

$e \cdot x^n = -x(x-1)^n$, $f \cdot x^n = -x(x+1)^n$, $h \cdot x^n = 2x^{n+1}$

or (Denote $\pi_a : V \to V$; $g(x) \mapsto g(x+a)$ and x as the scalar operator)

$e \rightsquigarrow -x \circ \pi_{-1}$, $f \rightsquigarrow -x \circ \pi_{+1}$, $h \longrightarrow 2x$

Then V is a rep which doesn't even have weight spaces.

II. Standard cyclic modules (Highest weight modules)

Def. $v^+ \in V_\lambda$: a maximal vector (of weight λ) if $\forall x \in N$, $x \cdot v^+ = 0$

V is standard cyclic if $V = \mathcal{U}(L) v^+$, λ is called the highest weight.

Ex. The L-mod $\mathcal{U}(L) \otimes_{\mathcal{U}(H \oplus N)} \mathbb{C} v^+ =: \Delta(\lambda)$ is called the Verma module corresponding to λ.

$\Delta(\lambda)$ is standard cyclic, $1 \otimes v^+$ is a maximal vector.

Thm. Simple finite dim mods are standard cyclic.

Pf. finite dim $\Rightarrow$ maximal vector exists. simplicity $\Rightarrow$ generated by any nonzero element

Thm. V is standard cyclic L-mod, with maximal vector $v^+ \in V_\lambda$. Let $\Phi^+ = \{\beta_1, \cdots, \beta_m\}$ Then

1. V is spanned by $y_{\beta_1}^{i_1} \cdots y_{\beta_m}^{i_m} v^+$, $i_j \in \mathbb{Z}_{\geq 0}$; in particular, $V = \bigoplus_{\mu \in H^*} V_\mu$
2. The weights of V are in $\lambda - \Lambda_r^+$
3. $\forall \mu \in H^*$, V_μ fin. dim and V_λ 1-dim.
4. $\forall W \subset V$, $W = \bigoplus_{\mu \in H^*} V_\mu \cap W$
5. V is indecomposible, with a unique maximal submod and a corresponding unique irr quotient
6. Every homom image of V is also standard cyclic of weight λ.

pf. 1. By PBW, $\mathcal{U}(L) = \mathcal{U}(N^-)\,\mathcal{U}(H)\,\mathcal{U}(N)$.

2. $y_{\beta_1}^{i_1} \cdots y_{\beta_m}^{i_m} v^+ \in V_{\lambda - \sum_{j=1}^m i_j \beta_j}$

3. $\forall \alpha \in \Lambda_r^+$, there are only finite number of the sum $\sum k_j \alpha_j$ equals α.

4. $\forall w \in W \subset V$, $w = v_{\mu_1} + v_{\mu_2} + \cdots + v_{\mu_n}$. WLOG, let $v_{\mu_j} \notin W$, $\forall j$. $(n>1)$

Let h s.t. $\mu_1(h) \neq \mu_2(h)$, Then

$$(h - \mu_1(h))\, w = (\mu_2 - \mu_1)(h)\, v_{\mu_2} + \cdots + (\mu_n - \mu_1)(h)\, v_{\mu_n} \in W$$

The sum of $(h - \mu_1(h))w$ is strictly shorter than w. After $n-1$ steps, we can get $v_{\mu_n} \in W$ which is a confliction.

5. If $W \cap V_\lambda \neq 0$, then $W = V$.

6. Let $\varphi : V \to W$, $\varphi(V) = \varphi(\mathcal{U}(L) v^+) = \mathcal{U}(L)\,\varphi(v^+)$ and $\varphi(v^+)$ is still a maximal vector.

Cor. V standard cyclic, the maximal vector is unique up to scalar. iff V is irred.

IV. Some properties of Verma modules

Thm. $\Delta(\lambda)$ has a unique maximal submod. Thus, it has a unique simple quotient.

pf. Let M be the sum of all proper submods of $\Delta(\lambda)$, Since $1 \otimes V_\lambda \notin M$, $M \neq \Delta(\lambda)$, Therefore, M is the unique maximal submod.

Moreover, $\Delta(\lambda)/M =: L(\lambda)$ is the unique simple quotient.

Thm. Each homom between Verma mods is injective.

pf. Let $\varphi : \Delta(\lambda) \to \Delta(\mu)$, $\varphi(v_\lambda) = u v_\mu$, $u \in U(N^-)$.

$\forall a \in U(N^-)$, $\varphi(a v_\lambda) = a\varphi(v_\lambda) = a u v_\mu$.

Since $U(N^-)$ is a domain, $\varphi(a v_\lambda) = 0 \iff a = 0$.

Thm. $\Delta(\lambda)$ has a simple socle. $\quad$ $soc(M) := \sum_{N \subseteq M, N \text{ simple}} N$

pf. By sl_2-theory on each $\mathrm{span}\{x_{\alpha_i}, y_{\alpha_i}, h_{\alpha_i}\} =: sl_2^{(i)}$ (α_i is a simple root), $\Delta(\lambda)$ has finite many maximal vectors.

Since every submodule of $\Delta(\lambda)$ have at least one maximal vector, $\Delta(\lambda)$ has finite many submodules.

Thus, there exist at least one simple submod of $\Delta(\lambda)$. $\quad$ Indeed, the above is true for any h.w.m.

$\Delta(\lambda) \cong U(N^-)$ as $U(N^-)$-mods.

Any two nonzero $U(N^-)$-submod of $U(N^-)$ have a nonzero intersection, since $U(N^-)$ is a domain. $\quad (\forall I, J \lhd U(N^-),\ IJ \lhd U(N^-))$

Thus, any two simple submod of $\Delta(\lambda)$ have a nonzero intersection.

So $soc(\Delta(\lambda))$ is simple.

In fact, $\dim \mathrm{Hom}(\Delta(\lambda), \Delta(\mu)) = \begin{cases} 1, & \lambda \le \mu, \lambda \in W\mu \\ 0, & \text{otherwise} \end{cases}$

Cor. $\dim \mathrm{Hom}(\Delta(\lambda), \Delta(\mu)) \le 1$, for all $\lambda, \mu \in H^*$

pf. Any nonzero homom maps simple socle to simple socle.

By Schur's lemma, any homom is a scalar on the socle.

If $\varphi, \psi \in \mathrm{Hom}(\Delta(\lambda), \Delta(\mu))$, there are $a, b \in F$, s.t. $a\varphi + b\psi$ kills the socle. By injectivity, $a\varphi + b\psi = 0$. Thus, the dimension is 1.

IV. Standard cyclic modules of sl_2

For sl_2 and a fixed $\lambda \in \mathbb{C}$, $\Delta(\lambda) = U(sl_2) \otimes_{U(b)} \mathbb{C} v_\lambda$ has a basis $\{f^i \otimes v_\lambda : i \in \mathbb{Z}_{\ge 0}\}$.

$$\begin{tikzcd} \cdots \arrow[r, bend left, "3(\lambda-2)"] & f^2 \otimes v_\lambda \arrow[l, bend left] \arrow[r, bend left, "2(\lambda-1)"] \arrow[loop below, "(\lambda-4)"] & f \otimes v_\lambda \arrow[l, bend left] \arrow[r, bend left, "\lambda"] \arrow[loop below, "(\lambda-2)"] & 1 \otimes v_\lambda \arrow[l, bend left] \arrow[loop below, "(\lambda)"] \end{tikzcd}$$

$\square$ e

$\square$ f

$\square$ h

Prop. $\Delta(\lambda)$ is simple iff $\lambda \notin \mathbb{Z}_{\geq 0}$

"$\Rightarrow$" obv.

"$\Leftarrow$" For any nonzero $x \in \Delta(\lambda)$, applying e finite times gives $1 \otimes v_\lambda$.
Then $\Delta(\lambda)$ can be generated by any nonzero element in $\Delta(\lambda)$.

Rmk. • If $\lambda \notin \mathbb{Z}_{\geq 0}$, $\Delta(\lambda)$ has the unique maximal vector $1 \otimes v_\lambda$ (up to scalar)
By the corollary, $\Delta(\lambda)$ is simple.

$e f^{\lambda+1} = f^{\lambda+1} e - (\lambda+1) f^{\lambda}(\lambda 1 - h)$

• If $\lambda \in \mathbb{Z}_{\geq 0}$, $1 \otimes v_\lambda$, $f^{\lambda+1} \otimes v_\lambda$ are both maximal vectors.
Since the quotient $\Delta(\lambda) / \mathcal{U}(sl_2) \cdot f^{\lambda+1} \otimes v_\lambda$ is the finite dim h.w.m $L(\lambda)$,
$\mathcal{U}(sl_2) f^{\lambda+1} \otimes v_\lambda$ is the maximal submodule of $\Delta(\lambda)$.

V. The universal property of Verma modules. (Frobinus reciprocity)

Thm. For all highest weight mods V of weight λ, there exists a surjective homom $\phi: \Delta(\lambda) \to V$
that is, V is isomorphic to a quotient of $\Delta(\lambda)$.

Pf. $\Delta(\lambda) = \mathcal{U}(L) \underset{\mathcal{U}(H \oplus N)}{\otimes} \mathbb{C} v_\lambda = \mathcal{U}(N^-) \otimes_F \mathbb{C} v_\lambda$ (a free $\mathcal{U}(N^-)$-mod)

Define $\phi: y_{\beta_1}^{i_1} \cdots y_{\beta_m}^{i_m} \otimes_F v_\lambda \longmapsto y_{\beta_1}^{i_1} \cdots y_{\beta_m}^{i_m} \cdot v_\lambda$.

By the theorem above, ϕ is surj.

A Generalization of the universal property is the following:

Thm. Given an L-mod M and $\lambda \in H^*$, denote $K_\lambda(M)$ by the set
$$K_\lambda(M) := \{ v \in M ;\ N_+ v = 0 \text{ and } h v = \lambda(h) v,\ \forall h \in H \}$$
Then $\mathrm{Hom}_{\mathcal{U}(L)}(\Delta(\lambda), M) \cong K_\lambda(M)$ (as $\mathcal{U}(H \oplus N)$-mod)

Pf. $\mathrm{Hom}_{\mathcal{U}(L)}(\Delta(\lambda), M) = \mathrm{Hom}_{\mathcal{U}(L)}(\mathcal{U}(L) \underset{\mathcal{U}(H \oplus N)}{\otimes} \mathbb{C} v_\lambda, M) \cong \mathrm{Hom}_{\mathcal{U}(H \oplus N)}(\mathbb{C} v_\lambda, \mathrm{Hom}_{\mathcal{U}(L)}(\mathcal{U}(L), M))$

$\cong \mathrm{Hom}_{\mathcal{U}(H \oplus N)}(\mathbb{C} v_\lambda, M) \cong K_\lambda(M)$.

$\mathrm{Hom}_R(B \otimes_S A, C) \cong \mathrm{Hom}_S(A, \mathrm{Hom}_R(B, C))$

Rmk. Indeed, it is the Frobenius reciprocity (more well known in group rep theory):

$$\mathrm{Hom}_{\mathcal{U}(L)}(\mathrm{Ind}_{\mathcal{U}(H \oplus N)}^{\mathcal{U}(L)} \mathbb{C} v_\lambda, M) \cong \mathrm{Hom}_{\mathcal{U}(H \oplus N)}(\mathbb{C} v_\lambda, \mathrm{Res}_{\mathcal{U}(H \oplus N)}^{\mathcal{U}(L)} M)$$